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SAMPLED-DATA ANALYSIS IN
PARAMETER SPACE

Technical rept.

S.M. Seltzer
Technology Laboratory

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This report describes a technique for determining the stability and dynamic characteristics of a digital control system in terms of several selected system parameters. The method requires that the system characteristic equation be available in the complex z-domain. An example is provided to further elucidate the technique. It is this capability that makes the method more powerful than most design techniques (which describe stability in terms of only one variable parameter or gain).

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CONTENTS

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1. INTRODUCTION

The parameter space method provides an analytical tool developed for use in control system analysis and synthesis. Although not necessary, its application is facilitated by augmenting the analytical results with graphical portrayals in a selected multi-parameter space. The method requires that the control system be described by a characteristic equation which, for sampled-data or digital systems, may be expressed in the z-domain. The technique is based on the analysis and synthesis methods for linear and nonlinear control system design which are amply described in Siljak's excellent monograph on the subject.[1] In another work, Siljak describes the application of the technique to the analysis and synthesis of linear sampled-data control systems.[2]

Once the system characteristic equation has been obtained, the parameter plane method enables the designer to evaluate graphically the locations of roots of the equation. Hence, he may design the control system in terms of the chosen performance criteria; e.g. absolute stability, damping ratio, and setting time. He is able to see the effect on the characteristic equation roots of changing two adjustable parameters. Siljak further simplified the design by introducing Chebyshev functions into the equations, thereby putting them in a form that is amenable to their solution by a digital computer.

The method has been extended to portray the effect of varying the sampling period.[3,4] The extended method permits one to see the effect of the choice of values assigned to the sampling period on absolute and relative stability. Also, the recursive formulas shown therein are simpler in form than the Chebyshev functions of Reference 2. The resulting formulation is deliberately cast in a form that makes it particularly amenable to solution by a digital computer or a desk calculator, again emphasizing the interplay between analysis and computing machines. When portrayed graphically, the results show the dynamic relation between the selected parameters and the characteristic equation roots, as a function of the nondimensional independent argument, $\omega_n T$. Hence one readily can deduce the dynamic effect upon the system of various combinations of values of the selected parameters defining the parameter space.

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1. D.D. Siljak, *Nonlinear Systems*, Wiley, New York, 1969.
2. D.D. Siljak, Analysis and Synthesis of Feedback Control Systems in the Parameter Plane, Part II-Sampled Data Systems," *Transactions of the Institute of Electrical and Electronics Engineers, Part II Applications and Industry*, Vol. 83, November 1964, pp. 458-466.
3. S.M. Seltzer, "Sampled-Data Control System Design in the Parameter Plane," *Proceedings of the 8th Annual Allerton Conference Circuits and System Theory*, 1970, pp. 454-463.
4. S.M. Seltzer, "Enhancing Simulation Efficiency with Analytical Tools," *Computers and Electrical Engineering*, Vol. 2, 1975, pp. 35-44.

The history of the continuous-time domain version of the parameter plane technique is well-described with suitable references in Reference 1. Briefly summarizing that history, in 1876 I.A. Vishnegradsky of the Leningrad School of Theoretical and Applied Mechanics developed and used the first version of the parameter plane technique to portray system stability and transient characteristics of a third order system on a two-parameter plane. In 1949 Professor Yu I. Neimark of the Russian School of Automatic Control generalized Vishnegradsky's approach to permit the decomposition of a two-parameter domain (D) describing an nth order system into stable and unstable regions. The technique was called D-decomposition. During the period 1959-1966 Professor D. Mitrovic, founder of a Belgrade group of automatic control, extended the method to enable the analyst to relate the system's variable parameters to the system response, using the last two coefficients of an nth order characteristic equation. Beginning in 1964. Professor D.D. Siljak, then a student of Mitrovic's at the University of Belgrade, generalized the method and called it the Parameter Plane method. His method permitted the analyst to select an arbitrary pair of characteristic equation coefficients (or parameters appearing within the coefficients) and portray both graphically and analytically the dependence of the system response upon the selected parameters. The method was extended subsequently by Siljak and others to encompass a host of related problems. In 1967 Professor J. George modified the D-decomposition method to enable the portrayal of the absolute stability region in a multi-parameter space. George also showed how to portray contours of relative stability, as did Siljak. All of the foregoing work is carefully and completely referenced within Reference 1. In 1966 and subsequent years, Seltzer has applied the parameter space method to the design of missile, aircraft, and satellite controllers, including systems containing one or two nonlinearities; the analysis of the dynamic effects of the nonliner "Solid Friction" (Dahl) model for systems with ball bearings, such as control moment gyroscopes and reaction wheels; and the specification by the system designer of the dynamic structural flexibility constraints to the structural designer. Most of the work has appeared in the technical journals of the Institute of Electrical and Electronic Engineers (IEEE), the American Institute of Astronautics and Aeronautics (AIAA), the International Journal of Control, and the journal, Computers & Electrical Engineering. A portion of the history that has not been reported upon previously (with one exception to be noted) is the control system work conducted by the German rocket scientists in the early 1940's at Peenemunde. There, Dr. W. Haeussermann and others applied the D-decomposition technique to the design of the V-2 Rocket, following Dr. Haeussermann's earlier (pre-World War II) application to the control of an underwater torpedo. This work was not published in the open literature because of national security constraints. When the group came to the United States to work with the Army Ballistic Missile Agency (first at Ft. Bliss, Texas, then at Redstone Arsenal, Huntsville, Alabama), Dr. Haeussermann and his associates continued to

apply the method to US Army missiles (and later to NASA space vehicles). Again, national (this time, another nation!) security precluded publication in the open literature until 1957.[5]

In 1964 Professor Siljak published the first application of the parameter plane technique to sampled-data systems.[2] As mentioned above, this was extended in References 3 and 4. In 1971 Seltzer presented an algorithm for systematically solving the Popov Criterion applied to sampled-data systems.[6] Applications of these sampled-data parameter space techniques are found in References 3, 4 and 6.

2. ANALYTICAL PRELIMINARIES

The technique requires that the control system be described by a characteristic equation in the z-domain. Two adjustable parameters (k_0, k_1) are selected, and the characteristic equation (CE) is recast in terms of them; i.e.

$$CE = \sum_{j=0}^{n} \gamma_j z^j = 0, \tag{1}$$

$$\gamma_j = \gamma_j (d_j, k_o, k_1) \tag{2}$$

$$z = e^{Ts} = re^{i\theta}, \tag{3}$$

$$r = r(\zeta, \omega_n, T) = e^{-\zeta\omega_n T}, \tag{4}$$

$$B = B(\zeta, \omega_n, T) = \cos\theta = \cos(\omega_n T \sqrt{1-\zeta^2}) = \cos\omega T \tag{5}$$

where ζ, ω_n and T represent the damping ratio, natural frequency, and sampling period, respectively. The symbol d_i represents all system parameters other than k_o, k_1. To transform the characteristic equation from an nth order polynomial into an algebraic equation, z^j may be defined in terms of its real and imaginary parts, R_j and I_j:

$$z^j = R_j + i I_j . \tag{6}$$

5. W. Haeussermann, "Stability Areas of Missile Control Systems," *Jet Propulsion*, July 1957, pp. 787-795.
6. S.M. Seltzer, "An Algorithm for Solving the Popov Criterion Applied to Sampled-Data Systems," *Proceedings of the Third Southeastern Symposium on System Theory*, 5-6 April 1971, pp. G5-0 through G5-7.

It readily follows from Equation (3) that the values of R_j and I_j are

$$R_j = r^j \cos j\theta \tag{7R}$$

and

$$I_j = r^j \sin j\theta . \tag{7I}$$

While equations (7R) and (7I) are satisfactory for determining the values of R_j and I_j, a set of recursive formulas can be derived that are particularly amenable to implementation on a desktop calculator or digital computer. They are shown below as Equation (8) and are derived in the appendix.

$$X_{j+1} = 2rBX_j - r^2 X_{j-1} , \tag{8}$$

where X_j may be either R_j or I_j. Only two values of X_j (i.e., two values each of R_j and I_j) are needed to obtain iteratively, one at a time, all other values of R_j, I_j. These are obtained from the definition of z in Equation (3). When $j=0$, the value of z^j is unity and, from Equation (6),

$$z^0 = R_0 + i I_0 \tag{9}$$

or

$$1 = R_0 \tag{10R}$$

$$0 = I_0 . \tag{10I}$$

When $j=1$, the value of z^j is, in Euler form,

$$z^1 = z = r \cos \theta + ir \sin \theta$$

$$= R_1 + i I_1 . \tag{11}$$

Hence,

$$R_1 = rB \tag{12R}$$

and

$$I_1 = r\sqrt{1-B^2} \ .$$
(12I)

It will turn out to be useful in the sequel to observe from Equation (8) that the radical, $\sqrt{1-B^2}$, appears as a factor in each value of I_j. Solely for simplicity, it may be factored out by defining a new term, $I_j{}'$ as

$$I_j{}' = I_j/\sqrt{1-B^2} \ .$$
(13)

It may be observed that R_j and I_j are functions of r and B or of $\zeta, \omega_n,$ and T. If Equation (6) is substituted into Equation (1) and the real and imaginary parts of the resulting equation are separated, two simultaneous algebraic equations are obtained.

These two equations contain the adjustable parameters, or variables, k_0, k_1. Hence it may be observed that the two equations may be solved explicitly for k_0 and k_1 as functions of the other system parameters (d_j) and, in particular, as functions of the independent argument, $\omega_n T$. It is the latter observation that forms the basis of the parameter space method for determining stability regions and system dynamic response in terms of selected system parameters.

3. STABILITY DETERMINATION

The method involves the definition of stability boundaries on a multi-parameter (to include k_0, k_1) space. These boundaries are found from the pair of simultaneous algebraic equations that result from the following operations on the system characteristic equation written in terms of the complex variable, z. Equation (6) is substituted into characteristic Equation (1), and the resulting real and imaginary parts are separated and equated to zero. The two resulting algebraic equations may be solved for any two parameters (such as k_0 and k_1) within them.

There may be as many as four stability boundaries for any system described by characteristic Equation (1), although it is not necessary for all four to exist.

One stability boundary separates the stable complex conjugate pairs of roots from the unstable ones. It consists of a map of the unit circle from the complex z-plane onto a selected (such as k_0, k_1) parameter space. Initially one may consider the oft-occurring case where the

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coefficients γ_j of the powers of z in the characteristic equation are linear combinations of k_0, k_1, i.e.,

$$\gamma_j = a_j k_o + b_j k_1 + c_j , \tag{14}$$

where a_j, b_j, c_j represent all system parameters other than k_0, k_1. In this case the two simultaneous equations resulting from the real and imaginary, respectively, parts of the characteristic equation assume the form,

$$\text{Re} \left\{C.E.\right\} = A_1 a + B_1 b - C_1 = 0, \tag{15R}$$

$$\text{Im} \left\{C.E.\right\} = A_2 a + B_2 b - C_2 = 0, \tag{15I}$$

and may be solved for k_0 and k_1 readily by applying Cramer's Rule, yielding

$$k_o = \frac{B_1 C_2 - B_2 C_1}{J} \tag{16}$$

$$k_1 = \frac{A_2 C_1 - A_1 C_2}{J} , \tag{17}$$

$$J = A_1 B_2 - A_2 B_1 \tag{18}$$

$$A_1 = \sum_{j=0}^{n} a_j R_j, \quad B_1 = \sum_{j=0}^{n} b_j R_j, \quad C_1 = \sum_{j=0}^{n} c_j R_j,$$

$$A_2 = \sum_{j=0}^{n} a_j I_j, \quad B_2 = \sum_{j=0}^{n} b_j I_j, \quad C_2 = \sum_{j=0}^{n} c_j I_j, \tag{19}$$

where J represents the Jacobian associated with the pair of algebraic equations. To find the stability boundary in question, one merely sets ζ equal to zero in the definitions of R_j and I_j used in obtaining k_0 and k_1 in Equations (16) and (17). The result is a boundary in the k_0-k_1 parameter plane that is defined in terms of system parameters d_j [in the general case, see Equation (2)] or a_j, b_j, c_j in the special case of Equation (14) and the independent argument, $\omega_n T$. The independent argument is varied in value between zero and π (assuming that the

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system being designed possesses low-pass filter characteristics so that only the "primary strip" need be considered), thereby defining the boundary in terms of k_0 and k_1. If the coefficients a_j, b_j, and c_j contain exponential terms with T appearing in the exponents, then each exponent may be replaced by its power series and truncated according to the accuracy that is desired. If the two parameters k_0 and k_1 do not appear linearly as expressed in Equation (14), they may be solved for by substituting Equation (6) into the characteristic Equation (1) and separating the result into the real and imaginary parts. This still results in two simultaneous algebraic equations which may be solved for the two variables, k_0 and k_1, although not as handily as expressed in Equations (16) and (17).

The next two stability boundaries are those separating the stable real roots from the unstable ones. These comprise a mapping of the $z=1$ and $z=-1$ points from the z-plane onto the selected parameter space. These are found readily by substituting $z=1$ and $z=-1$, respectively, into the characteristic Equation (1). Each of the two resulting equations results in a definition of the two real root boundaries in the selected parameter space.

The fourth boundary is a mapping of the conditions that cause the two simultaneous algebraic equations (used to define the complex conjugate root boundary) to become dependent. In the case where k_0 and k_1 appear linearly as in Equation (14), this case is found by determining the conditions that cause the Jacobian of Equation (18) to become identically equal to zero.

For a linear sampled-data control system to be stable, it is necessary that all roots of the characteristic equation lie within the unit circle on the z-plane. If the system is low-pass in nature, the stable region is bounded by the semi-circle defined by the upper half of the unit circle (as discussed above) and the singularities associated with $z = \pm 1$ (as discussed above) and $J=0$ (as discussed above). The mapping of these boundaries onto the parameter space will bound the stable region, if one exists. That region is determined by applying a shading criterion or using a test point.[1] If the Jacobian is greater than zero, then the stable region (if it exists) lies to the left of the complex conjugate root boundary as $\omega_n T$ increases; the left side of the line is double cross-hatched to indicate a boundary associated with double, or complex conjugate roots. (If the Jacobian is less than zero, the stable region lies to the right). Single cross-hatching is used on the two contours associated with the real roots. The side of the contour on which to place the cross-hatching is determined by the requirement that cross-hatching be continuous, or on the same side, of the contours as the intersections corresponding to $z= +1$ and $z= -1$ are approached along either the complex root or real root stability boundary. In the unusual (but physically possible) singular case when $J=0$, the shading of the resulting boundary is determined in a similar manner, i.e., the side on which to

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shade the boundary is determined by the physical requirement that the number of stable roots must never become less than zero as one moves across a stability boundary. This $J=0$ case may arise for a particular frequency, ω, or for a particular combination of system parameters. The latter situation is included in the work concerning a spinning Skylab.[7]

4. SYSTEM DYNAMICS

Once the stable region, if it exists, has been determined in the selected parameter space, the dynamic or transient characteristics of the system can be specified in terms of the locations of the roots of the characteristic equation (pole placement). For the complex conjugate roots, these locations are defined in terms of damping ratio (ζ) and system natural frequency (ω_n). Contours of constant ζ are determined as functions of ω_n and T in precisely the same manner that the complex conjugate stability boundary was determined except that ζ is not set equal to zero.

The real root locations corresponding to values of z when θ equals $0°$ and $180°$ also may be plotted on the parameter plane by setting z equal to a positive or negative real constant, substituting that value into Equation (1), and solving for k_1 as a function of k_0. Each resulting contour corresponds to a location of a real root in the z-domain. When $z=+\alpha$

$$\sum_{j=0}^{n} \gamma_j \alpha^j = 0, \tag{20}$$

and when $z=-\alpha$,

$$\sum_{j=0}^{n/2} \gamma_{2j}\alpha^{2j} - \sum_{j=0}^{(n-2)/2} \gamma(2_j+1_j)\alpha^{(2j+1i)} = 0, \; n \text{ even} \tag{21}$$

$$\sum_{j=0}^{(n-1)/2} (\gamma_{2j}\alpha^{2j} - \gamma_{(2j+1)}\alpha^{(2j+1)} = 0, \; n \text{ odd} \tag{22}$$

where α is a positive real number. (The two associated stability boundaries may be found by setting α equal to unity). Values of α corresponding to desired real root locations may be substituted into Equations (20) through (22) to obtain parameter space contours corresponding to these locations.

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7. S.M. Seltzer, "Passive Stability of a Spinning Skylab." *Journal of Spacecraft and Rockets*, Vol. 9, No. 9, September 1972, pp. 651-655.

Now the dynamic effect of any chosen design point in the parameter space may be specified in terms of ζ, ω_n, α, and T. The analytical technique developed permits the designer to observe the effect of simultaneously changing three control parameters and the sampling period. Most existing conventional techniques permit the observation of the effect of changing only one control parameter and don't show the effect of various sampling periods. If the designer is clever, sometimes the system parameters may be manipulated within the equations defining the stability and dynamic contours so that more than two parameters (such as k_0, k_1) may be used to define a parameter space. An example of a three-parameter space is provided in the sequel.

Sometimes it is specified that the setting time of the system be less than a prescribed value. This corresponds to requiring that the real part of the roots of the characteristic equation be less than a prescribed negative real constant. A boundary corresponding to this requirement can be drawn on the parameter plane by mapping a circle of constant radius (for a chosen values of ω_n, ζ, and T) from the z-plane onto the parameter plane.[7] Relations exist for estimating the maximum overshoot and peak time of transient response when it is valid to assume a second order system.[8] However, a simple estimate sometimes can be made merely by looking at the difference equation representing the system response, estimating when the overshoot will occur, and plotting a corresponding line on the parameter plane. Steady state responses may be found from the open loop transfer function and the assumed forcing functions in the conventional manner.

5. EXAMPLE

Consider the planar model that portrays controlled rigid body rotational dynamics. It is shown in block diagram form in *Figure 1*. The plant to be controlled is represented by the transfer functions,

$$G_4(s) = \Phi(s)/T_C(s) = 1/J_v s \tag{23}$$

and

$$G_5(s) = \Phi(s)/\Phi(s) = 1/s \tag{24}$$

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8. B.C. Kuo, *Analysis and Synthesis of Sampled-Data Control Systems*, Prentice-Hall, New Jersey, 1963.
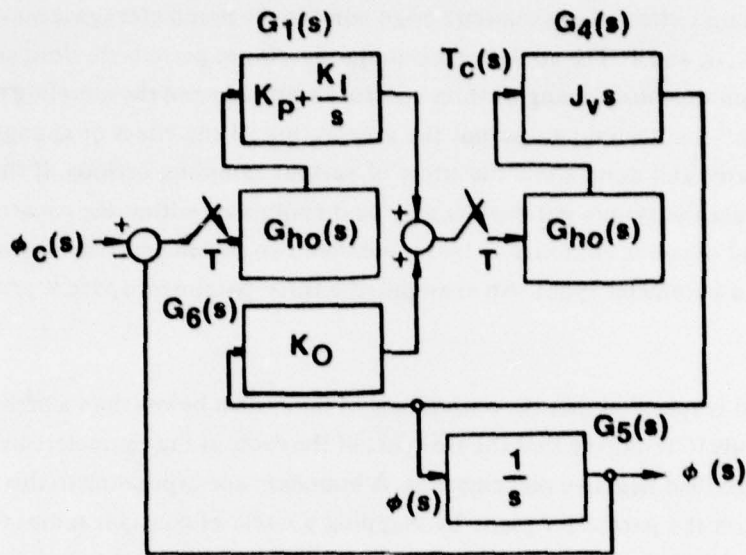
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Figure 1. Block diagram of controlled rigid body motion.

where s represents the Laplace operator, ϕ represents the rigid body attitude [in the time domain, $\phi(t)$; in the Laplace domain, $\phi(s)$], T_c represents the commanded torque, and the overdot represents the derivative with respect to time. The two sensors that measure attitude (ϕ) and attitude rates ($\dot{\phi}$) are assumed to be perfect with unity transfer functions. The on-board digital controller develops the commanded control torque from the input states ϕ and $\dot{\phi}$ and the commanded attitude signal ϕ_c; $G_{ho}(s)$ represents a zero-order hold in the computer:

$$Gh_o(s) = (1-e^{-Ts})/s, \tag{25}$$

where T is the sample period of the on-board digital computer. The PID control algorithm is represented by $G_1(s)$ and $G_6(s)$

$$G_1(s) = K_P + K_I/s \tag{26}$$

where K_P and K_I are the position and integral feedback gains and

$$G_6(s) = K_D \tag{27}$$

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where K_D is the derivative feedback gain (more commonly called the rate gain). The third order characteristic equation is Equation (1) with n=3, where the coefficients of z^i are in the form of Equation (14), i.e.,

$$\gamma_0 = (a - b + 2c - 1), \tag{28}$$

$$\gamma_1 = (-2a + 2c + 3), \tag{29}$$

$$\gamma_2 = (a + b - 3), \tag{30}$$

$$\gamma_3 = 1. \tag{31}$$

Modified gains a, b, c are defined as

$$a \equiv K_R T/J_v, \tag{32}$$

$$b \equiv K_p T^2/2J_v, \tag{33}$$

$$c \equiv K_I T^3/4J_v, \tag{34}$$

The stability boundaries are presented in terms of a,b,c, and $\omega_n T$ by observing that the roots (in this case three) of a characteristic equation representing a stable linear sampled-data control system must all lie within the unit circle in the z-plane. It is assumed that the system possesses low-pass filter characteristics so that only the primary strip (corresponding to $0 \leqslant \theta \leqslant \pi$) need be considered. The stable region may be defined by mapping its three boundaries from the z-plane onto the a, b, c parameter space by first considering parameters a, b to be k_0, k_1 in Equation (14). The boundary at $z = +1$ is found by substituting that value for z into the C.E., yielding the stability boundary

$$c = 0 \tag{35}$$

The boundary at $z = -1$ is found by substituting that value for z into the C.E., yielding

$$a = 2. \tag{36}$$

The complex conjugate root stability boundary may be found by setting z equal to $\cos \omega_n T + i \sin \omega_n T$ in the C.E. However, at this point it is convenient to use the recursive formulas of Equation (8) and transform the C.E. into two algebraic equations by separating the real and

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imaginary parts. If they are solved for a and b and the associated Jacobian J, one obtains as the complex conjugate root stability boundary,

$$a = [(1 + B)/(1 - B)] \, c + (1 - B), \tag{37}$$

$$b = c + (1 - B), \tag{38}$$

and

$$J = -4 \, (1 - B) < 0 \; \forall \; 0 < \Theta < \pi. \tag{39}$$

Examination of Equation (39) reveals that the stability boundary associated with the singular case $J=0$ only occurs when $B=1$, which is the already considered $z=1$ case. Now the stability boundaries of Equations (35), (36), (37) and (38) can be plotted on a three-dimensional plot with axes a, b, c. A sketch of these boundaries is shown in *Figure 2*. The stability region is found by applying the "cross-hatching" rules or by using one or more test points (known to be stable, to lie on a stability boundary, or to be unstable). If $J>0$ the stable region (if it exists) lies to the left of the complex conjugate root stability boundary of Equations (37) and (38), as θ (or $\omega_n T$) increases; the left side of the line is double cross-hatched to indicate a boundary
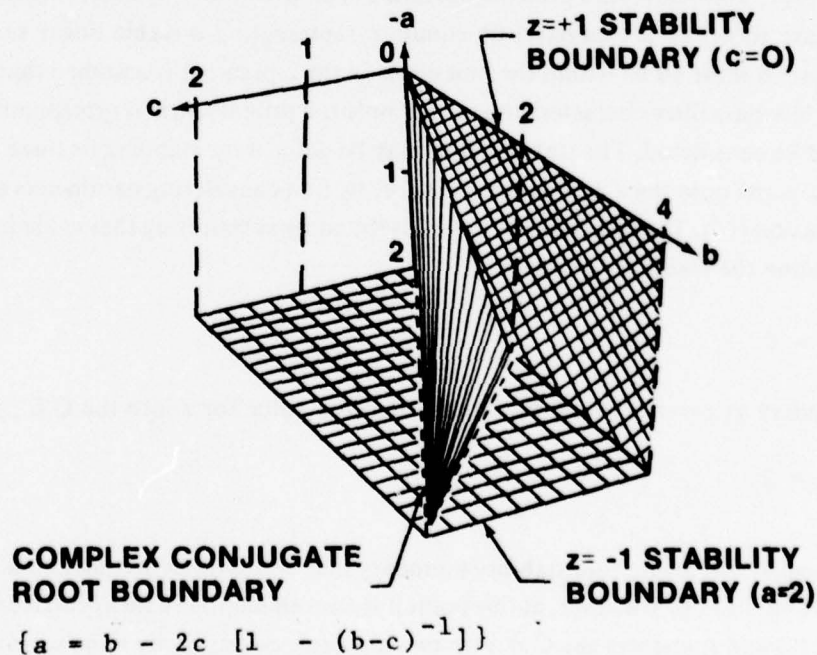

COMPLEX CONJUGATE ROOT BOUNDARY

$$\{a = b - 2c \, [1 - (b-c)^{-1}]\}$$

Figure 2. Stability boundaries in 3-dimensions.

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associated with double, or complex conjugate, roots. If $J<0$, as in this case, the stable region lies to the right and the right side of the line is double cross-hatched. Single cross-hatching is used on the contours of Equations (35) and (36) associated with single root boundaries. The side of the contour on which to place the cross-hatching is determined by the requirement that cross-hatching be continuous, or on the same side, of the contours as the intersections corresponding to $z=+1$ $(\theta=0)$ and $z=-1$ $(\theta=\pi)$ are approached along either the complex root or real root stability boundary. Using this criterion, the stable region is found to be bounded by the $c=0$ plane, the $a=2$ plane, and the curve specified by Equations (37) and (38). The latter may be solved for a as a function of b, resulting in the simple expression

$$a=b \ -2c \ \left[1-(b-c)^{-1}\right].\tag{40}$$

The limiting values for a and b may be found by letting θ approach 0 and π, i.e., $z=\pm 1$. For the case of $\theta \to 0$, represent B by the series expansion

$$B = \cos \theta = 1-(\theta^2/2!)+(\theta^4/4!)-(\theta^6/6!)+\ldots\tag{41}$$

Then using Equations (37), (38), and (41), one obtains

$$\lim_{\theta \to 0} b = c\tag{42}$$

$$\lim_{\theta \to 0} a \ . \ = \lim_{\theta \to 0} c \ \left[(4/\theta^2)-(2/3)+(\theta^2/36)-\ldots\right] \to \infty\tag{43}$$

For $c=0$, Equation (42) still holds (recognize $k_I=0$ is really the case one is interested in). Then, using Equations (34) and (37), one obtains

$$\lim_{\theta \to 0} a = \lim_{\theta \to 0} (k_I\theta^3/4J_v\omega^3)\frac{\left[2-(\theta^2/2!)+(\theta^4/4!)-\cdots\right]}{(\theta^2/2!)-(\theta^4/4!)+\cdots}$$

$$= \lim (k_I\theta/2J_v\omega^3)\left[2-(2\theta^2/6)+(\theta^4/72)-\cdots\right]=0\tag{44}$$

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For the limiting case of $\theta \rightarrow \pi$:

$$\lim_{\Theta \rightarrow \pi} a = \lim_{\Theta \rightarrow \pi} \left\{ [(1+B)c/(1-B)]+(j-B) \right\} = 2 \tag{45}$$

and

$$\lim_{\Theta \rightarrow \pi} b = \lim_{\Theta \rightarrow \pi} (c+1-B) = c+2 \tag{46}$$

The results of the foregoing are sketched graphically in two dimensions on *Figure 3*, for the cases where c = 0 and when c $\neq$ 0.

The design technique employs the pole placement approach. It is based on the premise that the dynamic behavior of the system is related closely to the location of the roots of its associated characteristic equation. The method shows both analytically and graphically the direct correlation between these roots and the control gains and sample period of the controller. The design technique then involves the specification of the characteristic equation root locations and the subsequent determination of the control system gains and sample period needed to attain these locations. The control system designer then must determine the system response resulting from using these numerical values and assess its adequacy. If it is not adequate, he usually relies on his experience to relocate the roots to improve the response in the manner desired for his particular system (i.e., faster settling time, lower peak overshoot, etc.). It is assumed that one wishes the pair of complex conjugate poles of the C.E. to dominate the dynamic response of the system. This response will then be modified by locating the third (real) root.

First consider the pair of complex conjugate roots. One may use Equations (16) - (19), substituting a, b for k_0, k_1, respectively [also using Equations (28) through (31)]. Upon examining the nature of the equations for a, b, one sees that they are functions of r, B, and c, or (finally) functions of ζ, θ, c. For each value of ζ, b versus a may be determined analytically and plotted graphically as a function of θ for a given value of c. This may be repeated for a number of values of c. Now the value of θ may be selected (for selected values of c) that best meets whatever design criteria one chooses. In effect, selecting θ and c will also choose the real root location. This may be shown by designating the real root of the C.E. as z_R, where

$$z_R \equiv \delta = e^{-\sigma_R T} \tag{47}$$

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NOTE: NUMBERS IN PARENTHESES ARE VALUES OF $\omega_n T$.

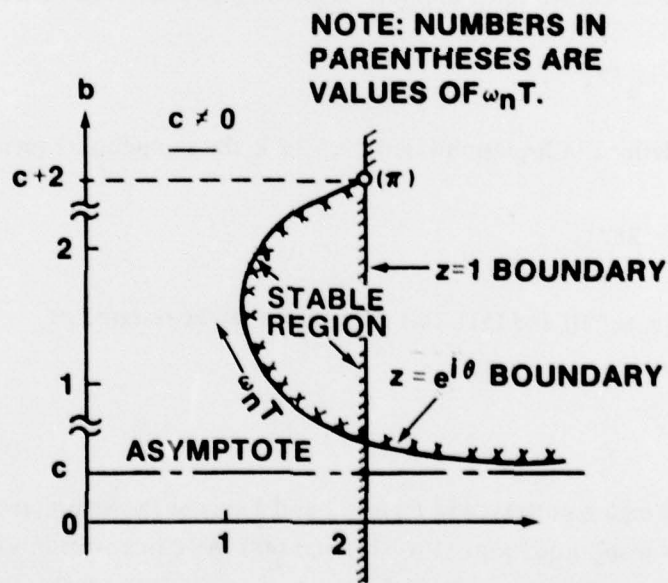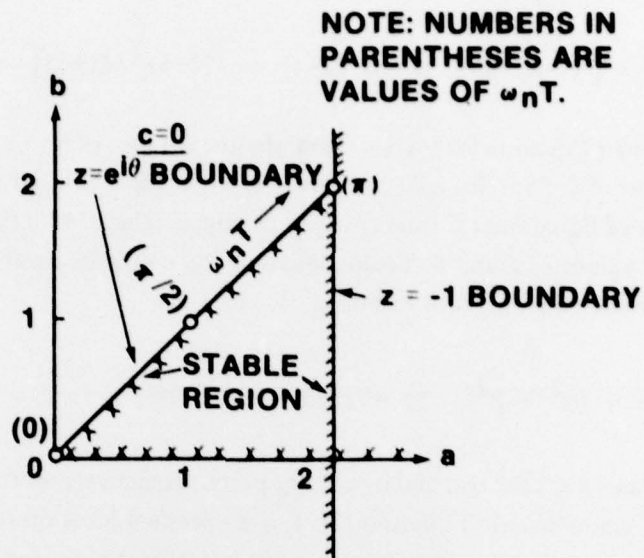**NOTE: NUMBERS IN PARENTHESES ARE VALUES OF $\omega_n T$.**

Figure 3. Stability boundaries in 2-dimensions.

and σ_R represents the damping factor. One may substitute Equation (47) into C.E. (1) and solve for b, obtaining

$$b = \left[(1-\delta)/1+\delta) \right] a + \left[2/(1-\delta) \ c - (1-\delta)^2/(1+\delta) \right] \tag{48}$$

One sees that Equation (48) yields a straight line contour of b vs. a for a given value of c and a given value of δ. Thus, for a given value of c, where the δ-contour of Equation (48) crosses the ζ-contour of Equations (37) and (38), one obtains a value of θ for the given value of c and those specified values of ζ and δ. From Equation (1), c may be determined as a function of the characteristic equation root locations:

$$c = (\alpha^2 + \beta^2 - 2\alpha + 1)(1 - \delta)/4, \tag{49}$$

where α and β are the real and imaginary parts, respectively, of the pair of complex conjugate roots of characteristic Equation (1). For a specified location (α, β) of the pair of complex conjugate roots, a value for c is established by setting a value for δ (location of the real root).

An additional constraint is imposed by Shannon's Sampling Theorem:

$$\omega < \omega_s/2, \tag{50}$$

where ω is defined in Equation (5) and $\omega_s/2\pi$ is the sampling frequency,

$$\omega_s = 2\pi/T. \tag{51}$$

From Equations (50) and (51), this constraint may be restated as

$$\Theta < \pi . \tag{52}$$

The stable region portrayed in *Figures 2* and *3* may be shown for several selected values of c on *Figure 4* using Equations (16), (17) and (48). As c increases in value, the stable region shrinks in size and the ζ contours lie further to the right. When c=0.6, the $\zeta = 1/\sqrt{2}$ contour lies entirely to the right (and outside) of the stable region. Finally when c$\geq$2, the stable region disappears entirely (for a geometric explanation, see *Figure 2*).
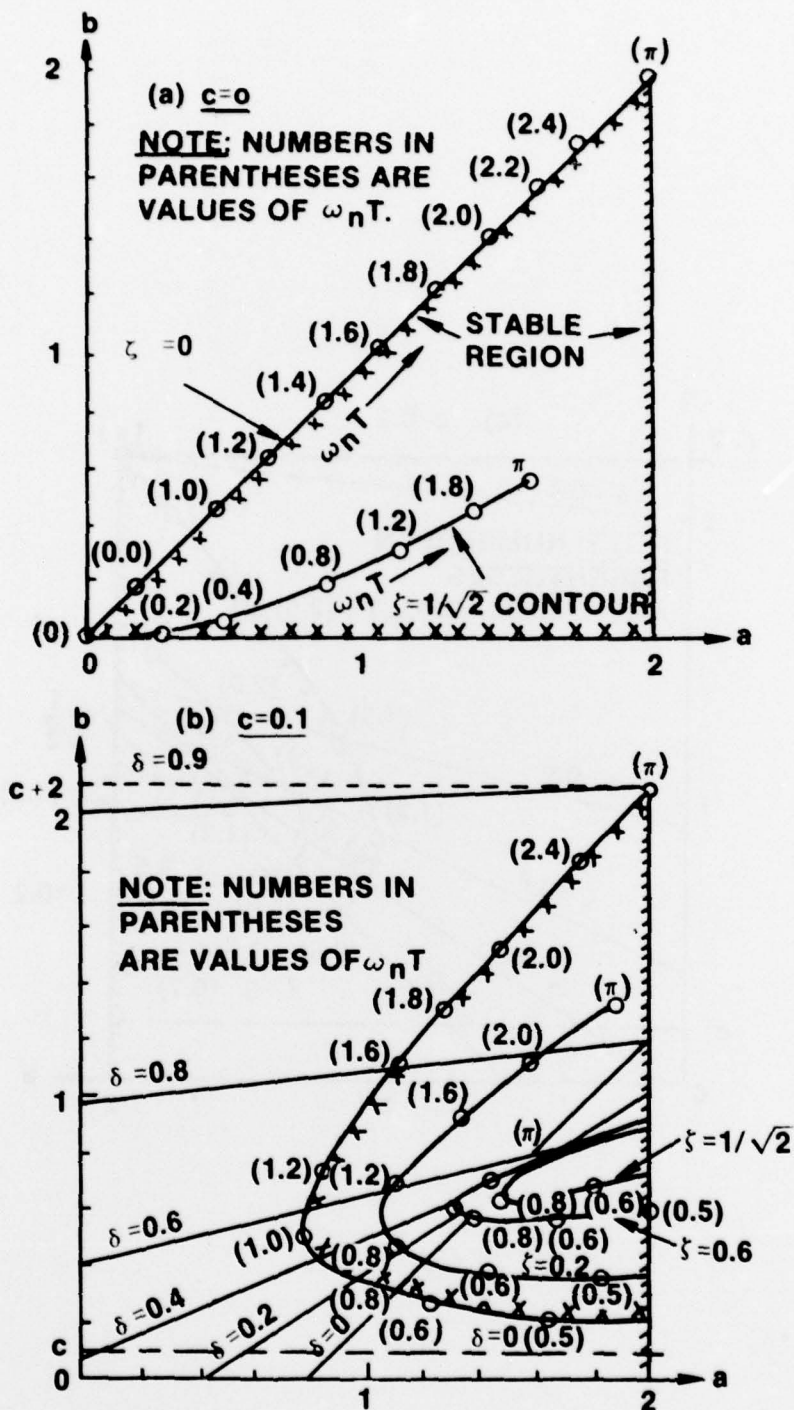
Figure 4. Parameter plane plots portraying root locations (c=0, 0.1, 0.2).
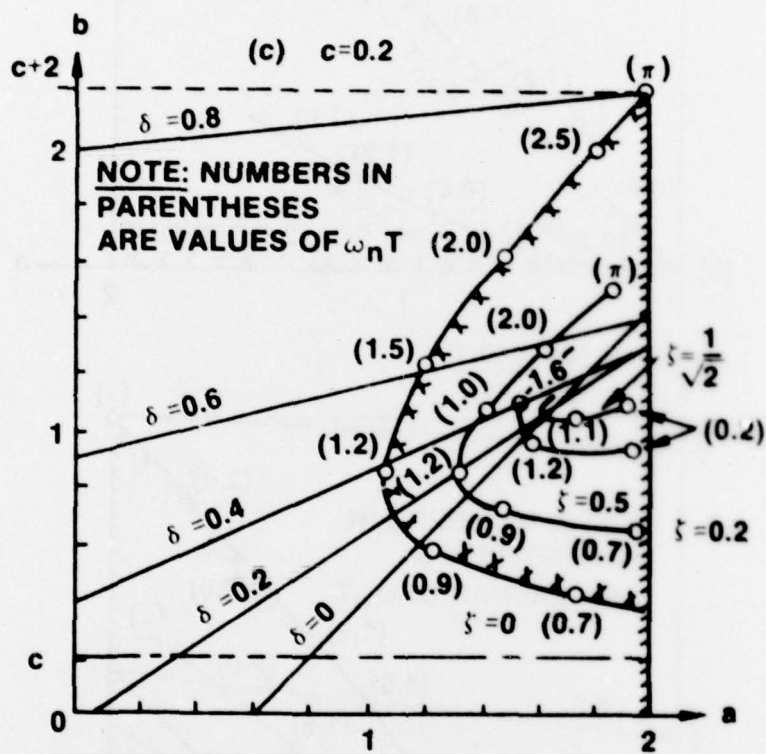
19

Figure 4. (Concluded).

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The application of the foregoing technique will now be summarized.

Based on desired system response characteristics, tentatively select locations for the three roots of the characteristic equation. Although the zeros of the system characteristic equation also affect the dynamics, placement of the poles will dominantly affect the dynamics.

- A root location criteria might be to select numerical values for ζ and ω_n.

- Determine computer design constraints. One might be to select as large a value of T as possible to minimize on-board memory capacity requirements.

- Then qualitatively determine how much integral gain is desired for the types of disturbance inputs expected. This gives an indication of the value of c to use.

Using the value of c tentatively established in the above paragraph, plot the stability boundaries in the b vs a parameter plane, using Equations (36) through (38). Then plot the desired ζ-contour (as a function of argument $\omega_n T$) on the b vs. a parameter plane, using Equations (16) and (17).

Using various values of δ, plot δ-contours on the same plot, using Equation (48).

Find an intersection of a δ-contour which gives a desired value for $\omega_n T$ (and hence T, since ω_n has already been specified) and δ. In general, if $\delta \rightarrow 0$, its effects on the system dynamics is small compared to the effect of the pair of complex conjugate poles (placed by $\omega_n T$, ζ).

Check the response of the system using the values of a, b, c, and $\omega_n T$ (and hence K_p, K_I, K_D, and T) associated with the selected intersection. If it is unsatisfactory, reiterate the above procedure, selecting another value of c.

If the design procedure were merely to specify the three root locations (such as in terms of α, β, δ, $\omega_n T$, δ), iteration would not be required. Unfortunately, in design practice one must exercise engineering judgement in selecting the three root locations, giving rise to iterative design procedures such as the one outlined above. However, another possibility does exist. If the real root is placed near the origin of the z-plane ($\delta \rightarrow 0$), and if the pair of complex conjugate roots are located near the unit circle in the z-plane (i.e., specifying ζ, and $\omega_n T$), the latter two roots will dominate the system response. Then the already developed tools for specifying the system response (such as determining explicitly the maximum overshoot and the settling time) of second-order systems may be applied handily. The foregoing procedure

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may be amplified through the application of realistic numerical values for the system parameters.

Assume a digital on-board controller, and further assume that it is desired to have a controller natural frequency (ω_n) of six rad/s, and that the moment of inertia (J_v) about the single axis is 4.3×10^4 kg.m^2. Assume that integral control is desired to drive to zero the effect of constant input disturbances. This means a non-zero value of c is desired. If a value of ζ of $1/\sqrt{2}$ is selected, and if the value of δ is desired to be kept as close to zero as possible, it is implied from *Figure 4c* that, for $\delta > 0$, no intersections of the ζ-contour occur when $c > 0.2$. Hence, a value of $c = 0.1$ is chosen. The smallest value of δ whose corresponding δ-line intersects the ζ-contour, with a reasonable factor of safety, is $\delta = 0.4$. The value of $\omega_n T = 1.2$ is on the ζ-contour at the intersection, yielding a value of 0.20s for T. Corresponding values of a and b are 1.41 and 0.68, respectively. This gives a set of gain values of 1.14×10^7 N·m/rad, 1.67×10^8 N·m/rad·s, and 2.35×10^6 N·m·s/rad for K_P, K_I, and K_D, respectively. If response studies show that a larger value of integral gain is deemed necessary, a larger value of c (and consequently a smaller value of ζ) would be selected.

6. CONCLUSIONS

An analytical method for portraying stability regions in a selected parameter space has been shown for a digital system. The method requires that the system characteristic equation be available and expressed in the complex z-domain. It also is possible to apply pole placement to obtain desired dynamic characteristics using this modified parameter space technique. The advantage of the technique over existing classical sampled-data methods is that the stability and dynamic response characteristics are expressed in terms of several (rather than merely one) selected parameters. Also, the sampling period, T, need not be expressed numerically before the design technique begins, giving the system designer one more degree of freedom.

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APPENDIX
DERIVATION OF RECURSIVE RELATIONS

It is possible to obtain algebraic recursive relations for the real and imaginary parts of the complex variable, z, when it is raised to the jth power (j is a possible integer). From Equation (3), z is defined as the vector (or complex variable),

$$z = e^{Ts} = r\ e^{i\theta} = r\ \cos\ \theta\ +i\ r\ \sin\ \theta. \tag{A1}$$

Using Equation (5) in Equation (A1), i.e., letting $B = \cos\ \theta$, leads to

$$z = rB + i\ r\ \sqrt{1-B^2}. \tag{A2}$$

The real and imaginary parts of z^j may be defined as R_j and I_j, respectively, i.e.,

$$z^j = R_j + i\ I_j\ . \tag{A3}$$

If $j=1$, Equation (A3) becomes identical with Equations (A1) and (A2), leading to

$$R_1 = rB \tag{A4R}$$

and

$$I_1 = r\ \sqrt{1-B^2}\ . \tag{A4I}$$

If $j=0$, then Equation (A3) becomes

$$z^0 = 1 = R_0 + i\ I_0\ , \tag{A5}$$

or

$$R_0 = 1 \tag{A6R}$$

and

$$I_0 = 0. \tag{A6I}$$

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If one squares the left and right hand sides of Equation (A1), one may obtain

$$z^2 = r^2 e^{i2\theta} = r^2 (\cos 2\theta + i \sin 2\theta)$$

$$= r^2 (2 \cos^2\theta - 1) + i2 r^2 \sin \theta \cos \theta$$

$$= 2 r^2 \cos \theta (\cos \theta + i \sin \theta) - r^2$$

$$= 2rBz - r^2. \tag{A7}$$

One may now multiply each term of Equation (A7) by z^{k-1} to yield

$$z^{k+1} = 2rBz^k - r^2 z^{k-1} . \tag{A8}$$

Substitution of Equation (A3) into Equation (A8), letting $j = k-1$, k, and $k+1$, leads to

$$R_{k+1} + i I_{k+1} = 2rB(R_k + i I_k) - r^2 (R_{k-1} + i I_{k-1}) . \tag{A9}$$

Separately equating the real and imaginary parts, respectively, of Equation (A9) leads to the two equations,

$$R_{k+1} = 2rBR_k - r^2 R_{k-1} \tag{A10R}$$

and

$$I_{k+1} = 2rBI_k - r^2 I_{k-1} . \tag{A10I}$$

One may observe the identical forms of Equations (A10) and write them both as a single equation,

$$X_{k+1} = 2rBX_k - r^2 X_{k-1} , \tag{A11}$$

where X_K may be used to represent either R_k or I_k. Of course the dummy index k may be exchanged with j, yielding Equation (8) of the text. If one knows the two values X_k and X_{k-1} of Equation (A11), one may determine the third value X_{k+1}. Since two sets of values of X_k are

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known, i.e., for k = 0 and k = 1 [see Equations (A4) and (A6)], as many values of X_k as are needed for the particular problem at hand can be determined recursively from Equation (A11).

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